The Separation Axioms and their interaction with Huge cardinals (and stronger)

Lantze Vongkorad

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Abstract

We expand and develop even more examples and properties in our previous paper on Counterexamples in Topology to focus on Separation Axioms more specifically.

1 Introduction and Overview

In our previous paper on Counterexamples in Topology (aptly named "Counterexamples in Topology generated by Large Cardinals, part I"), we gave about 4 examples of Large Cardinals generating interesting or bizarre results when conjoined with specific Large Cardinals. They are:

1. Homogeneity of spaces
2. Stone-Cech Compactification
3. Hausdorff-ness of the Strong Ultrafilter Topology
4. The Either-Or Topology

Separation axioms were not investigated too much in our previous paper, therefore ours will. Separation axioms and their interaction with Logic, particularly Set Theory, are not widely explored in the literature, other than brief applications and interesting properties such as genericity (and Forcing as a whole), absoluteness, and Martin's axiom. We also place much more emphasis on:

1. Topological consistency results that can be proved using a Large Cardinal,
2. and Topological hypothesis/assertions that imply (the consistency of) certain Large Cardinals.

Remark 1. The exploration of Normality in this paper is and will be limited, given normality's low intersection with cardinal arithmetic.

aligning with previous literature on Large Cardinals and Topology. To maintain the novelty of results in this paper, we will be exploring Large Cardinals stronger or equal to Huge cardinals.

2 Separation Axioms "themselves" and Large Cardinals

2.1 Rank-into-ranks

The only example we prove with regards to Separation Axioms in our previous work is of Hausdorff-ness of the Strong Ultrafilter Topology, but this is very specific. We proved that (3) fails under Vopenka's Principle.

Question 2.1. Is Hausdorff-ness in general preserved under Vopenka's Principle or weaker axiom?

Remark 2. When we say "preserved" or "holds under" a Large Cardinal, typically we mean Topological properties that are "as usual" even with a Large Cardinal assertion.

Most arguments and proofs dealing with (3) often deal with some property of the Strong Ultrafilter Topology first, typically via Ultrafilters and Embeddings, and then showing that said property is preserved via Ultrafilters and Embeddings to show that Hausdorff-ness is still "possible". The proof of Question 2.1 will of course rely on "Hausdorff-ness itself".

We start off with rank-into-rank axioms, one of the strongest Large Cardinal axioms not known to be inconsistent with AC. This is somewhat cheating, however, as rank-into-rank axioms are a slightly "metamathematical"[1] Large Cardinal axiom.

Theorem 2.2. *Hausdorff-ness of a Topological space is preserved under I1.*

Proof. We use induction "on" Topological spaces belonging to a specific V_κ, $\kappa < \lambda + 1$. "Preserved" in this context means if a Topological space X is Hausdorff in a specific V_{κ_n}, then it will be Hausdorff in a specific $V_{\kappa_{n+n}}$ (we assume that sets of V_κ-s are well-ordered, in that given $\kappa < \lambda+1$, $\langle V_\kappa : \kappa < \lambda+1\rangle$ is well-ordered[2]). Suppose that $X \in V_{\kappa_1}$, X a Hausdorff Topological space. Let $f : V_{\kappa_1} \to V_{\kappa_1}$, f is an injection. But Hausdorff-ness is preserved under injections. Continue for all $\langle V_\kappa : \kappa < \lambda + 1\rangle$. □

Theorem 2.3. *Hausdorff-ness of a Topological space is preserved under I2 and I3.*

Remark 3. Most properties dealing with Rank-into-rank axioms are on the functions and embeddings "in" the rank-into-ranks themselves rather than any properties "intrinsic" to said functions and embeddings nor the V_κ-levels. For example, from (Laver 1992) [3], one can generate a free algebra by means of j from $\varepsilon_\lambda = \{j : V_\lambda \prec V_\lambda\}$.

This raises the question: *what are some topological properties of functions formed from* $\varepsilon_\lambda = \{j : V_\lambda \prec V_\lambda\}$ *itself and other "purely from I0" sets?*

[1] Meta-set-theoretic or even meta-universal would be a better term.
[2] This is inspired by the well-order of λ when working with rank-into-rank embeddings.

Theorem 2.4. *Fix λ and let $\varepsilon_\lambda = \{j : V_\lambda \prec V_\lambda\}$. Let X be a set formed by embeddings of the form of j (admit them to X unrestrictedly). The topology on X formed by $\{(j,k) : j * k = j^+\}$ is open is Hausdorff.*

Note that Theorem 2.4 comes from the fact that if $j, k : V_\kappa \prec V_\kappa$, then $j^+(k) : V_\kappa \prec V_\kappa$, and '$*$' in this context means $j^+(k)$. And j^+ in the context of the *application* relation between embeddings refers to j^{++}. We also set the elementary embeddings here to Δ_0 (I3 holds).

Proof. Define a "point" in the set X as an embedding of the form of j. Define a "neighborhood" of a point in X as the set of embeddings $\{l_\kappa : j \subset dom(l_n), \forall \kappa < \lambda\}$, in which $l : V_\kappa \prec V_\kappa$. Let j_0 and j_1 be distinct embeddings. One needs to show that l_{0_κ} and l_{1_κ} (the "neighborhoods" of j_0 and j_1) also disjoint, in that $\{l_{0_\kappa}(j_0) : j_0 \subset dom(l_0), \forall \kappa < \lambda\}$ and $\{l_{1_\kappa}(j_1) : j_1 \subset dom(l_1), \forall \kappa < \lambda\}$. Proceed via nesting of lengths of sequences of the form $l, l(l), l(l(l)), ...$ and let $\langle \kappa_n : n \in \omega \rangle$. Then, via repeating $l(l^n)$ on embeddings like j_0 and j_1, we show that nested $l(l^n)$ are disjoint, thus two $\{l_{0_\kappa}(j_0) : j_0 \subset dom(l_0), \forall \kappa < \lambda\}$ and $\{l_{1_\kappa}(j_1) : j_1 \subset dom(l_1), \forall \kappa < \lambda\}$ are disjoint. □

Not even under I1 (in which the elementary embeddings set to Σ_n) is it the case that X (in Theorem 2.4) is Hausdorff. X appears to have very little structure.

Chris Good showed that Measurable cardinals can be used to prove several results about lattices and ultrafilters on topologies. For one, "there exists an infinite amount of measurable cardinals" implies that a finite lattice is isomorphic to the interval between two T_3 topologies on some set iff it is distributive. These lattices are formed from the set of all possible topologies on a set X. His original question was whether there is a characterization of finite lattices (formed from X) that are isomorphic to intervals between Hausdorff spaces. Measurable cardinals were eventually picked to answer this question. But given that we can form sequences of Measurable cardinals from Rank-into-rank axioms, it is no surprise that we can strengthen Good's original claims to Rank-into-ranks.

Theorem 2.5. *Given I3, one can generate a free ultrafilter $\Pi_{S_i < \omega} A$ on a set A via $\langle S_\alpha : \alpha < crit(j) \rangle$.*

In many of Good's theorems, they are applied to finite lattices under measurable cardinals. This can be extended to $> \aleph_0$ lattices however, under Rank-into-ranks. As a remark, one can create infinite *sequences* of measurables with Rank-into-ranks.

Theorem 2.6. *Given I1, one can freely choose topologies "on" the critical point of I1's embedding.*

Proof. "topologies on the critical point" means $\{(X, \tau) : X \in P(\kappa_0), \kappa_1 \in j(X)\}$, with τ obviously satisfying the axioms for a topology. □

Theorem 2.7. *Given such topologies from Theorem 2.6, if, say, X is Hausdorff, X is isomorphic to a countable lattice formed from* $\langle S_\alpha : \alpha < crit(j)\rangle$. *(I3)*

Proof. Let R be a well-founded relation on V_δ. Index members of the said lattice L (using a relation P) as $\{a_i : i \in n\}$ satisfying $i \neq j$ implying $a_i \neq a_j$. As X is Hausdorff, choose 2 separate points x and y in X, "represented" via the relation P "in" $\{a_i : i \in n\}$. Then there exists disjoint neighborhoods of x and y which are then represented by filters in L. □

We can even extend Theorem 2.7 to I1 and Normal spaces.

2.2 Huge Cardinals

We shift ourselves to Huge cardinals.

Theorem 2.8. *Continuous functions* $f : X \to Y$ *in* $j^n(\kappa)M$ *are restricted (with regards to the co-domain). (in* $j^n(\kappa)M \subset M$*), given* $j^n(\kappa)M$ *denoting the class of all sequences of length* j^n *whose elements are in* M.

Proof. Suppose that there is a continuous function in $j^n(\kappa)M$. Also, we can assume that the cardinality of said continuous function's range (in which we denote by Y) be $\leq j^n$. We can suppose that f, the cont. function in $j^n(\kappa)M$ is "transformed" n times (iterations, $j(f), j(j(f)), j^n(f)$). This results in a separate sequence of crit. points of $j(f), j(j(f)), j^n(f)$ represented as

$$\{\kappa^n : \text{critical point of } j^n(f)\}.$$

But as f need not be under any critical point, it must be the case that a function's image (in Y) is restricted ($\leq$) under $\{\kappa^n : \text{critical point of } j^n(f)\}$. □

Remark 4. Originally, Theorem 2.5's formulation was that T_6 (perfectly normal Hausdorff) spaces could not exist under Huge cardinals, but the proof using $\{\kappa^n : \text{critical point of } j^n(f)\}$ with f as a continuous function which precisely separates quickly fell apart and is now replaced with restrictions of continuous functions in general.

Chang's conjecture, proved by Chang in 1963, essentially states that every model of type (ω_2, ω_1) for a countable language has an elementary submodel of type (ω_1, ω). A model is of type (α, β) if it is of cardinality α and a unary relation is represented by a subset of cardinality β. Several variants and implications of Chang's conjecture are present in General Topology. Huge cardinals interact with Chang's conjecture in a huge way (pun intended); a much stronger version of Chang's conjecture (that $(\aleph_{\omega+1}, \aleph_\omega) \to (\aleph_1, \aleph_0)$)'s consistency follows from that of a 2-huge cardinal. To remark, an arguably weaker version of Chang's conjecture (that $(\omega_3, \omega_2) \to (\omega_2, \omega_1)$ also follows from an n-huge cardinal.

An elegant consequence of Chang's conjecture in General Topology explored by Peter Nyikos is that it is consistent[3] to have $(\kappa^+, \kappa) \to (\aleph_1, \aleph_0)$ for all $n < \omega$

[3] It also implies.

with a locally compact Hausdorff space X such that $d(X) < \aleph_\omega$ and $P(X)$ (in which $P(X)$ means cardinal function) s.t. $d(X) = \kappa$ and $P(X) > \kappa$ to have a separable subspace Y where $P(Y)$ is uncountable (Nyikos), in which $X \in (\aleph_{n+1}, \aleph_n)$ and $Y \in (\aleph_1, \aleph_0)$. To add, if X is compact, then this also holds for $P(X) = t(X)$. Therefore, one may ask if it is consistent under a very large cardinal, particularly a measurable cardinal. Also, one could easily replace "compact" with "Hausdorff" or "normal", but we still need to keep a cardinal invariant in our clause.

Question 2.9. Is it possible to use Nyikos's assumption and Chang's conjecture to show that limits of nets of a locally compact top. space X are not unique for $|X| > \aleph_1$?

If Question 2.9 is answered in the positive, this implies that:

2-huge cardinal $\Longrightarrow$ consistency of Chang's conjecture $\Longrightarrow$ Nyikos's assumption $\Longrightarrow$

non-Hausdorff-ness of spaces X s.t. $|X| > \aleph_1$.

Proof Sketch. We show that Nyikos's assumption and Chang's conjecture imply non-Hausdorff-ness of spaces X s.t. $|X| > \aleph_1$.[4] Also, use $(\aleph_{n+1}, \aleph_n) \to (\aleph_1, \aleph_0)$, in which $n < \omega_1$. Let a net N of X be such that $|N| \leq |Y|, Y \subset X, d(Y) > \aleph_1$, and $N \subset Y$ itself.

First of all, in defining N in this way (especially the cardinality of N), we have that it is consistent to have $d(N) \leq \aleph_1$, if X is compact. And as $|N| \leq |Y|$, $|N| \leq \aleph_2$. We can even extend this to other cardinal functions. Note that limits of nets in X are also cardinal functions. Because of this, simply substitute κ in Nyikos's assumption with $\aleph_{\omega_1+1}$, and $P(X)$ with limits of nets in X. But given the replacement of $P(X)$ with limits of arbitrary limits of nets in X, we have that limits of arbitrary nets in X are "homogenized". Limits of arbitrary nets in X are homogenized to $\aleph_1$. □

This implies that spaces $> \aleph_1$ are able to "remain" Hausdorff even when Chang's conjecture is added.

Remark 5. Initially, the proof of Question 2.9 was focused on spaces of all cardinalities, but only the $> \kappa$ case is true.

Per ([5]), he constructs a model of ZFC in which Huge cardinals are consistent. As a core lemma inspired by Tall,

Lemma 2.10. *Given a discrete collection $\mathscr{Y}$ of subsets of a topological space X, with each point in $\mathscr{Y}$ being of weight $< \aleph_1$ and of cardinality $< \aleph_1$, there*

[4] One can even treat this theorem as a special case of Nyikos's assumption.

is a subspace X' of X generated by $\{Z_Y\}_{Y \in \mathscr{Y}}$ (in which $\{Z_Y\}_{Y \in \mathscr{Y}}$ denotes Z_Y the space generated from choosing points from various $Y \in \mathscr{Y}$) such that X is Hausdorff if and only if X' is Hausdorff.

Proof Sketch. Fix a basis $\mathscr{B}_Y$ for each $Y \in \mathscr{Y}$. Also, fix a neighborhood system for each $y \in Y$. The rest of the proof goes as how Tall proves his Lemma 6. □

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